

Two new multi-component BKP hierarchies ^{*†}

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Abstract

We firstly propose two kinds of new multi-component BKP (mcBKP) hierarchy based on the eigenfunction symmetry reduction and nonstandard reduction, respectively. The first one contains two types of BKP equation with self-consistent sources which Lax representations are presented. The two mcBKP hierarchies both admit reductions to the k -constrained BKP hierarchy and to integrable (1+1)-dimensional hierarchy with self-consistent sources, which include two types of SK equation with self-consistent sources and of bi-directional SK equations with self-consistent sources.

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1. Introduction

The multi-component KP (mcKP) hierarchy given in [1] contains many physically relevant nonlinear integrable systems, such as Davey-Stewartson equation, two-dimensional Toda lattice and three-wave resonant interaction ones, and attracts a lot of interests from both physical and mathematical points of view [1-8]. Another kind of multi-component KP equation is the so-called KP equation with self-consistent sources, which was initiated by V.K. Mel'nikov [9-11]. The first type of KP equation with self-consistent sources (KPSCS) arises in some physical modes describing the interaction of long and short wave [8-10,12], and the second type of KPSCS is presented in [8,11,13]. However, little attention has been paid to the multi-component BKP hierarchy. Though the first type of the BKP equation with self-consistent sources (BKPSCS) is constructed by source generating method [14], the Lax representation for the first type of BKPSCS and the second type of the BKPSCS have not been investigated yet.

It is known that the Lax equation of KP hierarchy is given by [15]

$$L_{t_n} = [B_n, L] \quad (1.1)$$

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where

$$L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \cdots \quad (1.2)$$

is pseudo-differential operator, ∂ denotes ∂/∂_x , u_i , $i = 1, 2, \dots$, are functions in infinitely many variables $t = (t_1, t_2, t_3, \dots)$ with $t_1 = x$, and $B_n = L_+^n$ stands for the differential part of L^n .

Owing to the commutativity of ∂_{t_n} flows, we obtain zero-curvature equations of KP hierarchy

$$B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0 \quad (1.3)$$

Eigenfunction Φ (adjoint eigenfunction Φ^*) satisfy the linear evolution equations

$$\Phi_{t_n} = B_n(\Phi) \quad (\Phi_{t_n}^* = -B_n^*(\Phi^*)) \quad (1.4)$$

The compatibility condition of (1.4) is exactly (1.3).

The BKP hierarchy is obtained from the KP hierarchy by imposing the condition

$$L^* \partial + \partial L = 0 \quad (1.5)$$

The formula (1.5) implies the vanishing of the even time variables (i.e., $t_2 = t_4 = \dots = 0$) and of the constant terms $B_n, n = 3, 5, \dots$, as well as that $u_2 = -u_1'$, $u_4 = -2u_3' + u_1^{(3)}$, \dots , and $\Phi^* = \Phi'$ for n odd. Taking $k = 3, n = 5$, (1.3) and (1.5) gives rise to the BKP equation

$$u_{t_5} + \frac{1}{9}u^{(5)} - \frac{5}{9}u_{t_3}^{(2)} + \frac{5}{3}uu^{(3)} + \frac{5}{3}u'u^{(2)} - \frac{5}{3}uu_{t_3} + 5u^2u' - \frac{5}{3}u'\partial_x^{-1}u_{t_3} - \frac{5}{9}\partial_x^{-1}u_{t_3t_3} = 0 \quad (1.6)$$

where we use the notation $u^{(i)} = \frac{\partial^i}{\partial x^i}$, $u' = \frac{\partial}{\partial x}$ in this paper.

In this paper, following the idea in [8] and using the eigenfunction symmetry constraint, we firstly introduce a new type of Lax equations which consist of the new time τ_k -flow and the evolutions of wave functions. Under the evolutions of wave functions, the commutativity of the evolutions of τ_k -flow and t_n -flow gives rise to the first kind of new mcBKP hierarchy. This hierarchy enables us to obtain the first and the second types of BKPHSCS and their related Lax representations directly. This implies that the new mcBKP hierarchy can be regarded as BKP hierarchy with self-consistent sources (BKPHSCS). Moreover, this new mcBKP hierarchy can be reduced to two integrable equation hierarchies: a (1+1)-dimensional soliton equation hierarchy with self-consistent sources and the k -constrained BKP hierarchy (k -BKPH), which contain the first type and the second type of SK equation with self-consistent sources and of bi-direction SK equation with self-consistent sources, respectively. Similar to the construction of the first kind of mcBKP hierarchy, we can also construct the second kind of mcBKP hierarchy based on nonstandard reduction to obtain some new (2+1)-dimensional soliton equation with self-consistent sources. It is noted that the second kind of mcBKP hierarchy just as the first kind also admits the n -reduction and the k -constraint, which lead to some new (1+1)-dimensional soliton equations with self-consistent sources. Thus, these two mcBKP hierarchies provide an effective way to find (1+1)-dimensional and (2+1)-dimensional soliton equations with self-consistent sources as well as their Lax representations. Our paper is organized as follows. In section 2, we construct the first kind of new mcBKP hierarchy based on eigenfunction symmetry constraint and show that it contains the first and the second types

of BKPSCS. In section 3, the mcBKP hierarchy is reduced to a (1+1)-dimensional soliton hierarchy with self-consistent source and the k -constrained BKP hierarchy, respectively. In section 4, the second kind of new mcBKP hierarchy is also proposed based on nonstandard reduction. In addition, the n -reduction and the k -constraint of it are also considered. In section 5, some conclusions are given.

2. The first kind of new mcBKP hierarchy

Following the idea in [8] and using the eigenfunction symmetry constraint for BKP hierarchy [16], we define \tilde{B}_k by

$$\tilde{B}_k = B_k + \sum_{i=1}^N (r_i \partial^{-1} q'_i - q_i \partial^{-1} r'_i) \quad (2.1)$$

where q_i, r_i satisfy (1.4). Then we may introduce a new Lax equation given by

$$L_{\tau_k} = [B_k + \sum_{i=1}^N (r_i \partial^{-1} q'_i - q_i \partial^{-1} r'_i), L] \quad (2.2a)$$

$$q_{i,t_n} = B_n(q_i), \quad r_{i,t_n} = B_n(r_i), \quad i = 1, \dots, N \quad (2.2b)$$

where n, k are odd.

Lemma 1 $[B_n, r \partial^{-1} q' - q \partial^{-1} r']_- = (r \partial^{-1} q' - q \partial^{-1} r')_{t_n}$

Proof: Set $B_n = \sum_{i=1}^n a_i \partial^i$ ($i \geq 1$). Then we have

$$\begin{aligned} [B_n, r \partial^{-1} q' - q \partial^{-1} r']_- &= \sum_{i=1}^n (a_i r^{(i)} \partial^{-1} q' - a_i q^{(i)} \partial^{-1} r') - \sum_{i=1}^n (r \partial^{-1} q' a_i \partial^i - q \partial^{-1} r' a_i \partial^i)_- \\ &= B_n(r) \partial^{-1} q' - B_n(q) \partial^{-1} r' - \sum_{i=1}^n (r \partial^{-1} q' a_i \partial^i - q \partial^{-1} r' a_i \partial^i)_- \end{aligned}$$

Applying integration by parts to the second term

$$\sum_{i=1}^n (r \partial^{-1} q' a_i \partial^i - q \partial^{-1} r' a_i \partial^i)_- = \dots = \sum_{i=1}^n (-1)^i [r \partial^{-1} (a_i q')^{(i)} - q \partial^{-1} (a_i r')^{(i)}] = r \partial^{-1} B_n^*(q') - q \partial^{-1} B_n^*(r')$$

Noticing the facts that $q^* = q', r^* = r', q_{t_n}^* = -B_n^*(q^*)$ and $r_{t_n}^* = -B_n^*(r^*)$, we can complete the proof immediately.

Theorem 1. The commutativity of (1.1) and (2.2a) under (2.2b) leads to the following first kind of new integrable multi-component BKP (mcBKP) hierarchy

$$B_{n,\tau_k} - (B_k + \sum_{i=1}^N (r_i \partial^{-1} q'_i - q_i \partial^{-1} r'_i))_{t_n} + [B_n, B_k + \sum_{i=1}^N (r_i \partial^{-1} q'_i - q_i \partial^{-1} r'_i)] = 0 \quad (2.3a)$$

or equivalently

$$\begin{aligned} B_{n,\tau_k} - B_{k,t_n} + [B_n, B_k] + \sum_{i=1}^N \{ [B_n, r_i \partial^{-1} q'_i - q_i \partial^{-1} r'_i] + B_n(q_i) \partial^{-1} r'_i \\ + q_i \partial^{-1} B'_n(r_i) - B_n(r_i) \partial^{-1} q'_i - r_i \partial^{-1} B'_n(q_i) \} = 0 \end{aligned} \quad (2.3a')$$

$$q_{i,t_n} = B_n(q_i), \quad r_{i,t_n} = B_n(r_i), \quad i = 1, \dots, N \quad (2.3b)$$

where n and k are odd. Under (2.3b), the Lax pair for (2.3a) is given by

$$\psi_{t_n} = B_n(\psi), \quad \psi_{\tau_k} = [B_k + \sum_{i=1}^N (r_i \partial^{-1} q'_i - q_i \partial^{-1} r'_i)](\psi) \quad (2.4)$$

Proof: We will show that under (2.3b), (1.1) and (2.2a) lead to (2.3a). For convenience, we assume $N = 1$ and denote q_1, r_1 by q, r . By (1.1), (2.2) and lemma 1, we have

$$\begin{aligned} B_{n,\tau_k} &= (L_{\tau_k}^n)_+ = [B_k + r \partial^{-1} q' - q \partial^{-1} r', L^n]_+ = [B_k + r \partial^{-1} q' - q \partial^{-1} r', L_+^n]_+ + [B_k + r \partial^{-1} q' - q \partial^{-1} r', L_-^n]_+ \\ &= [B_k + r \partial^{-1} q' - q \partial^{-1} r', L_+^n] - [B_k + r \partial^{-1} q' - q \partial^{-1} r', L_+^n]_- + [B_k, L_-^n]_+ \\ &= [B_k + r \partial^{-1} q' - q \partial^{-1} r', B_n] - [r \partial^{-1} q' - q \partial^{-1} r', B_n]_- + [B_n, L^k]_+ \\ &= [B_k + r \partial^{-1} q' - q \partial^{-1} r', B_n] + (r \partial^{-1} q' - q \partial^{-1} r')_{t_n} + (B_k)_{t_n} \\ &= [B_k + r \partial^{-1} q' - q \partial^{-1} r', B_n] + (B_k + r \partial^{-1} q' - q \partial^{-1} r')_{t_n} \end{aligned}$$

Remark 1. (2.3a') and (2.4) indicate that the new mcBKP hierarchy can be regarded as the BKP hierarchy with self-consistent sources and that it is Lax integrable.

Next we will list some examples in the first kind of mcBKP hierarchy.

Example 1 (The first type of BKPCS) For $n = 3, k = 5$, (2.3) with $u = u_1$ leads to the first type of the BKP equation with self-consistent sources ((2+1)-dimensional CDGKS equation with self-consistent sources)

$$\begin{aligned} u_{\tau_5} + \frac{1}{9} u^{(5)} - \frac{5}{9} u_{t_3}^{(2)} + \frac{5}{3} u u^{(3)} + \frac{5}{3} u' u^{(2)} - \frac{5}{3} u u_{t_3} + 5 u^2 u' - \frac{5}{3} u' \partial_x^{-1} u_{t_3} - \frac{5}{9} \partial_x^{-1} u_{t_3 t_3} + \sum_{i=1}^N (q_i^{(2)} r_i - r_i^{(2)} q_i) &= 0, \\ q_{i,t_3} = q_i^{(3)} + 3 u_1 q'_i, \quad r_{i,t_3} = r_i^{(3)} + 3 u_1 r'_i, \quad i = 1, \dots, N \end{aligned} \quad (2.5)$$

If we omit the source, we will obtain (2+1)-dimensional CDGKS equation [17].

The Lax pair of (2.5) is given by

$$\begin{aligned} \psi_{t_3} &= (\partial^3 + 3u\partial)(\psi), \\ \psi_{\tau_5} &= (\partial^5 + 5u\partial^3 + \frac{15}{2} u' \partial^2 + (\frac{5}{3} \partial_x^{-1} u_{t_3} + \frac{10}{3} u^{(2)} + 5u^2) \partial + \sum_{i=1}^N (r_i \partial^{-1} q'_i - q_i \partial^{-1} r'_i))(\psi) \end{aligned} \quad (2.6)$$

Example 2 (The second type of BKPCS) For $n = 5, k = 3$, (2.3) with $u_1 = u$ yields the second type of

BKP equation with self-consistent sources

$$\begin{aligned}
& u_{t_5} + \frac{1}{9}u^{(5)} - \frac{5}{9}u_{\tau_3}^{(2)} + \frac{5}{3}uu^{(3)} + \frac{5}{3}u' u^{(2)} - \frac{5}{3}uu_{\tau_3} + 5u^2u' - \frac{5}{3}u' \partial_x^{-1}u_{\tau_3} \\
& - \frac{5}{9}\partial_x^{-1}u_{\tau_3\tau_3} = \frac{1}{9}\sum_{i=1}^N [5q_i^{(3)}r_i' - 5r_i^{(3)}q_i' + 10q_i^{(4)}r_i - 10r_i^{(4)}q_i \\
& + 5(q_i'r_i - r_i'q_i)_{\tau_3} + 30uq_i^{(2)}r_i - 30ur_i^{(2)}q_i + 30u'q_i'r_i - 30u'r_i'q_i], \\
& q_{i,t_5} = q_i^{(5)} + 5uq_i^{(3)} + 5u'q_i^{(2)} + [\frac{5}{3}\partial_x^{-1}u_{\tau_3} + \frac{10}{3}u^{(2)} + 5u^2 + \frac{5}{3}\sum_{i=1}^N (q_i'r_i - q_i r_i')]q_i', \\
& r_{i,t_5} = r_i^{(5)} + 5ur_i^{(3)} + 5u'r_i^{(2)} + [\frac{5}{3}\partial_x^{-1}u_{\tau_3} + \frac{10}{3}u^{(2)} + 5u^2 + \frac{5}{3}\sum_{i=1}^N (q_i'r_i - q_i r_i')]r_i'
\end{aligned} \tag{2.7}$$

The Lax pair of (2.7) is given by

$$\begin{aligned}
\psi_{\tau_3} &= [\partial^3 + 3u\partial + \sum_{i=1}^N (r_i\partial^{-1}q_i' - q_i\partial^{-1}r_i')] (\psi), \\
\psi_{t_5} &= [\partial^5 + 5u\partial^3 + 5u'\partial^2 + (\frac{5}{3}\partial_x^{-1}u_{\tau_3} + \frac{10}{3}u^{(2)} + 5u^2 + \frac{5}{3}\sum_{i=1}^N (q_i'r_i - q_i r_i'))\partial] (\psi)
\end{aligned} \tag{2.8}$$

Remark 2 The first type of BKPCS (2.5) coincide with what obtained in [14] by source generating method. However the Lax representation for (2.5) and the second type of BKPCS have not been found before.

3. The n - reduction and k - constraint of (2.3)

3.1 The n - reduction of (2.3)

The n - reduction of (2.3) is given by [15]

$$L^n = B_n, \text{ or } L_-^n = 0 \tag{2.9}$$

which implies that

$$L_{t_n} = [B_n, L] = [L^n, L] = 0, \quad B_{k,t_n} = (L_+^k)_{t_n} = 0, \quad \text{and} \quad q_{i,t_n} = r_{i,t_n} = 0 \tag{2.10}$$

If q_i and r_i are wave function, they have to satisfy [15]

$$B_n(q_i) = L^n(q_i) = \lambda_i^n q_i, \quad B_n(r_i) = L^n(r_i) = \lambda_i^n r_i \tag{2.11}$$

So it is reasonable to impose the relation (2.11) in the n - reduction case. By using the Lemma 1 and (2.10), we can conclude that the constraint (2.9) is invariant under the τ_k - flow. Due to (2.10) and (2.11), one can drop t_n - dependency from (2.3) and get the following (1+1)-dimensional integrable hierarchy with self-consistent sources

$$\begin{aligned}
& B_{n,\tau_k} + [B_n, B_k + \sum_{i=1}^N (r_i\partial^{-1}q_i' - q_i\partial^{-1}r_i')] = 0, \\
& B_n(q_i) = \lambda_i^n q_i, \quad B_n(r_i) = \lambda_i^n r_i, \quad i = 1, \dots, N
\end{aligned} \tag{2.12}$$

with the Lax pair given by

$$\begin{aligned} B_n(\psi) &= \lambda^n \psi, \\ \psi_{\tau_k} &= [B_k + \sum_{i=1}^N (r_i \partial^{-1} q_i' - q_i \partial^{-1} r_i')](\psi) \end{aligned} \quad (2.13)$$

Example 3 (The first type of SKSCS) For $n = 3, k = 5$, (2.12) present the first type of SK equation with self-consistent sources (SKSCS)

$$\begin{aligned} u_{\tau_5} + \frac{1}{9}u^{(5)} + \frac{5}{3}u' u^{(2)} + \frac{5}{3}uu^{(3)} + 5u^2 u' + \sum_{i=1}^N (q_i^{(2)} r_i - q_i r_i^{(2)}) &= 0, \\ q_i^{(3)} + 3u q_i' &= \lambda_i^3 q_i \\ r_i^{(3)} + 3u r_i' &= \lambda_i^3 q_i, \quad i = 1, \dots, N \end{aligned} \quad (2.14)$$

(2.13) with $n = 3, k = 5$ leads to the Lax pair of (2.14)

$$\begin{aligned} (\partial^3 + 3u\partial)(\psi) &= \lambda^3 \psi, \\ \psi_{\tau_5} &= [\partial^5 + 5u\partial^3 + 5u' \partial^2 + (\frac{10}{3}u^{(2)} + 5u^2)\partial + \sum_{i=1}^N (r_i \partial^{-1} q_i' - q_i \partial^{-1} r_i')](\psi) \end{aligned} \quad (2.15)$$

Example 4 (The first type of bi-directional SKSCS) For $n = 5, k = 3$, (2.12) presents the first type of bi-directional SK equation with self-consistent sources (bi-directional SKSCS)

$$\begin{aligned} \frac{1}{9}u^{(5)} - \frac{5}{9}u_{\tau_3}^{(2)} + \frac{5}{3}uu^{(3)} + \frac{5}{3}u' u^{(2)} - \frac{5}{3}uu_{\tau_3} + 5u^2 u' - \frac{5}{3}u' \partial_x^{-1} u_{\tau_3} \\ - \frac{5}{9}\partial_x^{-1} u_{\tau_3 \tau_3} = \frac{1}{9} \sum_{i=1}^N [5q_i^{(3)} r_i' - 5r_i^{(3)} q_i' + 10q_i^{(4)} r_i - 10r_i^{(4)} q_i \\ + 5(q_i' r_i - r_i' q_i)_{\tau_3} + 30u q_i^{(2)} r_i - 30u r_i^{(2)} q_i + 30u' q_i' r_i - 30u' r_i' q_i], \\ q_i^{(5)} + 5u q_i^{(3)} + 5u' q_i^{(2)} + [\frac{5}{3}\partial_x^{-1} u_{\tau_3} + \frac{10}{3}u^{(2)} + 5u^2 + \frac{5}{3} \sum_{i=1}^N (q_i' r_i - q_i r_i')] q_i' &= \lambda_i^5 q_i, \\ r_i^{(5)} + 5u r_i^{(3)} + 5u' r_i^{(2)} + [\frac{5}{3}\partial_x^{-1} u_{\tau_3} + \frac{10}{3}u^{(2)} + 5u^2 + \frac{5}{3} \sum_{i=1}^N (q_i' r_i - q_i r_i')] r_i' &= \lambda_i^5 r_i, \quad i = 1, \dots, N \end{aligned} \quad (2.16)$$

with the Lax pair given by

$$\begin{aligned} \psi_{\tau_3} &= [\partial^3 + 3u\partial + \sum_{i=1}^N (r_i \partial^{-1} q_i' - q_i \partial^{-1} r_i')](\psi), \\ [\partial^5 + 5u\partial^3 + 5u' \partial^2 + (\frac{5}{3}\partial_x^{-1} u_{\tau_3} + \frac{10}{3}u^{(2)} + 5u^2 + \frac{5}{3} \sum_{i=1}^N (q_i' r_i - q_i r_i'))\partial](\psi) &= \lambda^5 \psi \end{aligned} \quad (2.17)$$

If we take $q_i = r_i = 0$, then (2.14) and (2.16) reduces to the SK equation and bi-directional SK equation [18].

3.2 The k - constraint of (2.3)

The k - constraint of (2.3) is given by [16]

$$L^k = B_k + \sum_{i=1}^N (r_i \partial^{-1} q_i' - q_i \partial^{-1} r_i') \quad (2.18)$$

It can be seen that (2.18) together with (2.2) lead to $L_{\tau_k} = 0$ and $B_{n,\tau_k} = 0$. Then (2.3) becomes k - constrained BKP hierarchy

$$\begin{aligned} (B_k + \sum_{i=1}^N (r_i \partial^{-1} q_i' - q_i \partial^{-1} r_i'))_{t_n} &= [(B_k + \sum_{i=1}^N (r_i \partial^{-1} q_i' - q_i \partial^{-1} r_i'))_+^{\frac{n}{k}}, B_k + \sum_{i=1}^N (r_i \partial^{-1} q_i' - q_i \partial^{-1} r_i')], \\ q_{i,t_n} &= (B_k + \sum_{i=1}^N (r_i \partial^{-1} q_i' - q_i \partial^{-1} r_i'))_+^{\frac{n}{k}}(q_i), r_{i,t_n} = (B_k + \sum_{i=1}^N (r_i \partial^{-1} q_i' - q_i \partial^{-1} r_i'))_+^{\frac{n}{k}}(r_i), \quad i = 1, \dots, N \end{aligned} \quad (2.19)$$

Example 5 (The second type of SKSCS) For $n = 5, k = 3$, (2.19) presents the second type of SK equation with self-consistent sources

$$\begin{aligned} u_{t_5} + \frac{1}{9}u^{(5)} + \frac{5}{3}uu^{(3)} + \frac{5}{3}u'u^{(2)} + 5u^2u' &= \frac{1}{9} \sum_{i=1}^N [5q_i^{(3)}r_i' - 5r_i^{(3)}q_i' + 10q_i^{(4)}r_i \\ &- 10r_i^{(4)}q_i + 30uq_i^{(2)}r_i - 30ur_i^{(2)}q_i + 30u'q_i'r_i - 30u'r_i'q_i], \\ q_{i,t_5} &= q_i^{(5)} + 5uq_i^{(3)} + 5u'q_i^{(2)} + [\frac{10}{3}u^{(2)} + 5u^2 + \frac{5}{3} \sum_{i=1}^N (q_i'r_i - q_i r_i')]q_i', \\ r_{i,t_5} &= r_i^{(5)} + 5ur_i^{(3)} + 5u'r_i^{(2)} + [\frac{10}{3}u^{(2)} + 5u^2 + \frac{5}{3} \sum_{i=1}^N (q_i'r_i - q_i r_i')]r_i', \quad i = 1, \dots, N \end{aligned} \quad (2.20)$$

Example 6 (The second type of bi-directional SKESCS) For $n = 3, k = 5$, (2.19) gives rise to the second type of bi-directional SK equation with self-consistent sources

$$\begin{aligned} \frac{1}{9}u^{(5)} - \frac{5}{9}u_{t_3}^{(2)} + \frac{5}{3}uu^{(3)} + \frac{5}{3}u'u^{(2)} - \frac{5}{3}uu_{t_3} + 5u^2u' - \frac{5}{3}u'\partial_x^{-1}u_{t_3}, \\ -\frac{5}{9}\partial_x^{-1}u_{t_3t_3} + \sum_{i=1}^N (q_i^{(2)}r_i - r_i^{(2)}q_i) &= 0, \\ q_{i,t_3} &= q_i^{(3)} + 3u_1q_i', \quad r_{i,t_3} = r_i^{(3)} + 3u_1r_i', \quad i = 1, \dots, N \end{aligned} \quad (2.21)$$

Next we will consider the special case of the first type of mBKP hierarchy (2.3). Then, starting from the first type one based on the symmetry eigenfunction constraint, we may obtain one mBKP hierarchy based on the nonstandard reduction. Noting that a constant is the eigenfunction for (1.4), (2.3) with $q_i = 1$ and $R_i = -r_i'$ yield the following mBKP hierarchy

$$\begin{aligned} B_{n,\tau_k} - (B_k + \sum_{i=1}^N \partial^{-1} R_i)_{t_n} + [B_n, B_k + \sum_{i=1}^N \partial^{-1} R_i] &= 0, \\ R_{t_n} &= -B_n^*(R) \end{aligned} \quad (2.22)$$

The Lax pair associated with (2.22) reads

$$\psi_{t_n} = B_n(\psi), \quad \psi_{\tau_k} = (B_k + \sum_{i=1}^N \partial^{-1} R_i)(\psi) \quad (2.23)$$

where k and k are both odd.

Example 7 For $n = 3, k = 5$, (2.22) gives rise to

$$u_{\tau_5} + \frac{1}{9}u^{(5)} - \frac{5}{9}u_{t_3}^{(2)} + \frac{5}{3}uu^{(3)} + \frac{5}{3}u' u^{(2)} - \frac{5}{3}uu_{t_3} + 5u^2u' - \frac{5}{3}u' \partial_x^{-1}u_{t_3} - \frac{5}{9}\partial_x^{-1}u_{t_3 t_3} + \sum_{i=1}^N R_i' = 0, \quad (2.24)$$

$$R_{i,t_3} = R_i^{(3)} + 3(uR_i)', \quad i = 1, \dots, N$$

The Lax pair of (2.24) is given by

$$\psi_{\tau_5} = [\partial^5 + 5u\partial^3 + 5u' \partial^2 + (\frac{5}{3}\partial_x^{-1}u_{t_3} + \frac{10}{3}u^{(2)} + 5u^2)\partial + \sum_{i=1}^N \partial^{-1}R_i](\psi), \quad (2.25)$$

$$\psi_{t_3} = (\partial^3 + 3u\partial)(\psi)$$

Example 8 For $n = 5, k = 3$, (2.22) presents

$$u_{t_5} + \frac{1}{9}u^{(5)} - \frac{5}{9}u_{\tau_3}^{(2)} + \frac{5}{3}uu^{(3)} + \frac{5}{3}u' u^{(2)} - \frac{5}{3}uu_{\tau_3} + 5u^2u' - \frac{5}{3}u' \partial_x^{-1}u_{\tau_3} - \frac{5}{9}\partial_x^{-1}u_{\tau_3 \tau_3} = \frac{1}{9} \sum_{i=1}^N [\frac{10}{9}R_i^{(3)} + \frac{10}{3}(uR_i)' + 5R_{i,\tau_3}], \quad (2.26)$$

$$R_{i,t_5} = R_i^{(5)} + 5(uR_i^{(2)})' + 5(u' R_i')' + (\frac{5}{3}\partial_x^{-1}u_{\tau_3} + \frac{10}{3}u^{(2)} + 5u^2 + \frac{5}{3} \sum_{i=1}^N R_i)R_i'$$

with the Lax pair given by

$$\psi_{\tau_3} = [\partial^3 + 3u\partial + \sum_{i=1}^N \partial^{-1}R_i](\psi), \quad (2.27)$$

$$\psi_{t_5} = [\partial^5 + 5u\partial^3 + 5u' \partial^2 + (\frac{5}{3}\partial_x^{-1}u_{\tau_3} + \frac{10}{3}u^{(2)} + 5u^2 + \frac{5}{3} \sum_{i=1}^N R_i)\partial](\psi)$$

The k -constraint of (2.22) is given by [19]

$$L^k = B_k + \partial^{-1}R \quad (2.28)$$

Combining (2.22) with (2.28), we have

$$(B_k + \partial^{-1}R)_{t_n} = [\{B_k + \partial^{-1}R\}_+^{\frac{n}{k}}, B_k + \partial^{-1}R], \quad (2.29)$$

$$R_{t_n} = -[(B_k + \partial^{-1}R)_+^{\frac{n}{k}}]^*(R), i = 1, \dots, N$$

Example 9 For $n = 3, k = 5$, we obtain 5-constrained equation from (2.29)

$$\frac{1}{9}u^{(5)} - \frac{5}{9}u_{t_3}^{(2)} + \frac{5}{3}uu^{(3)} + \frac{5}{3}u' u^{(2)} - \frac{5}{3}uu_{t_3} + 5u^2u' - \frac{5}{3}u' \partial_x^{-1}u_{t_3} - \frac{5}{9}\partial_x^{-1}u_{t_3 t_3} + \sum_{i=1}^N R_i' = 0, \quad (2.30)$$

$$R_{i,t_3} = R_i^{(3)} + 3(uR_i)'$$

Example 10 For $n = 5, k = 3$, we obtain 3-constrained equation from (2.29)

$$\begin{aligned} u_{t_5} &= -\frac{1}{9}u^{(5)} - \frac{5}{3}uu^{(3)} - \frac{5}{3}u'u^{(2)} - 5u^2u' + \frac{10}{9}R^{(3)} + \frac{10}{3}(uR)', \\ R_{t_5} &= R^{(5)} + 5[(uR)^{(3)} - (u'R)^{(2)} + (u^2R)'] + \frac{5}{3}(2u^{(2)}R + R^2) \end{aligned} \quad (2.31)$$

We notice that (2.31) coincide with what obtained in [19].

4. The second kind of new mcBKP hierarchy

In the previous section, the first kind of new mcBKP hierarchy is constructed based on the eigenfunction symmetry reduction. In fact, we find that we can construct the second kind of new mcBKP hierarchy by using its nonstandard reduction given by [20], which is completely different from the first kind.

It is show in [20] that the non-symmetry constraint

$$L^k = B_k + \sum_{i=1}^N (r_i \partial^{-1} q_i' + q_i \partial^{-1} r_i'), k \text{ is even} \quad (2.32)$$

reduces the BKP hierarchy to a (1+1)-dimensional integrable hierarchy. So as for the symmetry constraint, we may define a new Lax equation

$$L_{\tau_k} = [B_k + \sum_{i=1}^N (r_i \partial^{-1} q_i' + q_i \partial^{-1} r_i'), L] \quad (2.33a)$$

$$q_{i,t_n} = B_n(q_i), \quad r_{i,t_n} = B_n(r_i), \quad i = 1, \dots, N \quad (2.33b)$$

where n is odd and k is even.

In the exactly same way as for Lemma 1, we can find

Lemma 2. $[B_n, r \partial^{-1} q' + q \partial^{-1} r']_- = (r \partial^{-1} q' + q \partial^{-1} r')_{t_n}$

Theorem 2 (1.1) and (2.33) lead to the second kind of new integrable mcBKP hierarchy

$$B_{n,\tau_k} - (B_k + \sum_{i=1}^N (r_i \partial^{-1} q_i' + q_i \partial^{-1} r_i'))_{t_n} + [B_n, B_k + \sum_{i=1}^N (r_i \partial^{-1} q_i' + q_i \partial^{-1} r_i')] = 0 \quad (2.34a)$$

$$q_{i,t_n} = B_n(q_i), \quad r_{i,t_n} = B_n(r_i), \quad i = 1, \dots, N \quad (2.34b)$$

where n is odd and k is even. With the Lax pair for (2.34a) under (2.34b) given by

$$\begin{aligned} \psi_{\tau_k} &= (B_k + \sum_{i=1}^N (r_i \partial^{-1} q_i' + q_i \partial^{-1} r_i'))(\psi), \\ \psi_{t_n} &= B_n(\psi) \end{aligned} \quad (2.35)$$

Example 11 For $n = 3, k = 2$, (2.34) leads to the following new integrable (2+1)-dimensional equations

$$\begin{aligned} u_{\tau_2} + u^{(2)} + \sum_{i=1}^N (q_i r_i)^{(2)} &= 0, \\ -u_{t_3} + u^{(3)} + 3uu' + 3 \sum_{i=1}^N (q_i' r_i^{(2)} + r_i' q_i^{(2)}) &= 0, \\ q_{i,t_3} = q_i^{(3)} + 3uq_i', \quad r_{i,t_3} = r_i^{(3)} + 3ur_i', \quad i = 1, \dots, N \end{aligned} \quad (2.36)$$

Example 12 For $n = 3, k = 4$, (2.34) yields to another new integrable (2+1) dimensional equation

$$\begin{aligned} 3u_{\tau_4} + 2u_{t_3}' + u^{(4)} + 6(u')^2 + 6uu^{(2)} + 3 \sum_{i=1}^N (q_i r_i)^{(2)} &= 0, \\ \frac{2}{3}u_{t_3}^{(2)} - \frac{4}{3}\partial_x^{-1}u_{t_3 t_3} + \frac{2}{3}u^{(5)} + 12u' u^{(2)} + 12u' u^2 + 6uu^{(3)} + 6 \sum_{i=1}^N (q_i' r_i^{(2)} + r_i' q_i^{(2)}) &= 0, \\ q_{i,t_3} = q_i^{(3)} + 3uq_i', \quad r_{i,t_3} = r_i^{(3)} + 3ur_i', \quad i = 1, \dots, N \end{aligned} \quad (2.37)$$

Similar to the first kind of new mcBKP hierarchy, the second kind of mcBKP hierarchy also admits the n -reduction and the k -constraint given by $B_n = L^n$ and $L^k = B_k + \sum_{i=1}^N (r_i \partial^{-1} q_i' + q_i \partial^{-1} r_i')$, respectively.

Example 13 For $n = 3, k = 2$, the 2-constrained (2.34) is given by

$$q_{i,t_3} = q_i^{(3)} - 3q_i r_i q_i', \quad r_{i,t_3} = r_i^{(3)} - 3q_i r_i r_i' \quad (2.38)$$

Example 14 For $n = 3, k = 4$, the 4-constrained (2.34) is given by

$$\begin{aligned} u_{t_3} &= -\frac{1}{2}u^{(3)} - 3uu' - \frac{3}{4} \sum_{i=1}^N (q_i r_i)', \\ q_{i,t_3} &= q_i^{(3)} + 3uq_i', \quad r_{i,t_3} = r_i^{(3)} + 3ur_i', \quad i = 1, \dots, N \end{aligned} \quad (2.39)$$

It is noticed that (2.38) is the coupled mKdV equation which has been obtained in [20] and that (2.39) is a new KdV equation with self-consistent sources which is completely different from the two known KdV equations with self-consistent sources introduced by [8, 21-23].

Example 15 The 3-reduction of (2.34) for $n = 3, k = 2$ reads

$$\begin{aligned} u_{\tau_2} - \frac{3}{2}u^2 + \sum_{i=1}^N (q_i^{(2)} r_i + r_i^{(2)} q_i - q_i' r_i') &= 0, \\ q_i^{(3)} + 3uq_i' = \lambda_i^3 q_i, \quad r_i^{(3)} + 3ur_i' = \lambda_i^3 r_i, \quad i = 1, \dots, N \end{aligned} \quad (2.40)$$

Example 16 The another 3-reduction of (2.34) for $n = 3, k = 4$ reads

$$\begin{aligned} 3u_{\tau_4} + \frac{1}{3}u^{(4)} + 3(u')^2 - 4u^3 + 3 \sum_{i=1}^N (q_i r_i^{(2)} + q_i^{(2)} r_i) &= 0, \\ q_i^{(3)} + 3uq_i' = \lambda_i^3 q_i, \quad r_i^{(3)} + 3ur_i' = \lambda_i^3 r_i, \quad i = 1, \dots, N \end{aligned} \quad (2.41)$$

5. Conclusion

We firstly propose two new kinds of multi-component BKP hierarchy based on the eigenfunction symmetry reduction and nonstandard reduction, respectively. The first kind of mcBKP hierarchy includes two types of BKP equation with self-consistent sources. It admits reductions to the k - constrained BKP hierarchy containing the second type of $(1+1)$ -dimensional integrable soliton equation with self-consistent sources, and n - reduction of BKP hierarchy including the first type of $(1+1)$ -dimensional integrable soliton equation with self-consistent sources. Then we construct the second kind of mcBKP hierarchy based on nonstandard reduction to obtain some new integrable $(2+1)$ -dimensional soliton equation with self-consistent sources. It is noted that the second kind of mcBKP hierarchy also admits the n - reduction and k - constraint, which lead to some new integrable $(1+1)$ -dimensional soliton equation with self-consistent sources. Thus, these two mcBKP hierarchies provide an effective way to find $(1+1)$ -dimensional and $(2+1)$ -dimensional soliton equations with self-consistent sources as well as their Lax representations. Though the solution for the first type of BKPSCS was constructed by source generating method [14], the solution structure for the second type of BKPSCS has not been investigated yet. We will solve the integrable equation in the forthcoming paper.

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